p-ADIC PROPERTIES OF COEFFICIENTS OF CERTAIN HALF-INTEGRAL WEIGHT MODULAR FORMS

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ABSTRACT. In this paper, we study the parallel cases of Zagier's and Folsom-Ono's grids of weakly holomorphic (resp. weakly holomorphic and mock modular) forms of weights 3/2 and 1/2, investigating their p-adic properties under the action of Hecke operators.

1. INTRODUCTION AND STATEMENT OF RESULTS

In [14], Zagier builds two sets of interlocking weakly holomorphic modular forms of level 4 in Kohnen's plus space – one set of weight 3/2 and one of weight 1/2 – such that their coefficients not only form a grid, but also give the traces of singular moduli, the values of the j-function at CM-points. Zagier also uses these forms to reconstruct a theorem of Borcherds [4], which enables the computation of minimal polynomials of these singular moduli.

Inspired by Zagier's work, Duke and Jenkins explore in [9] other half-integral weight weakly holomorphic modular forms in Kohnen's plus space of level 4. In [6], Bringmann, Guerzhoy, and Kane continue the study of these forms by investigating their p-adic properties, using a lifting procedure developed by Duke and Jenkins to link back to Zagier's original forms in order to give a p-adic relation between half-integral weight weakly holomorphic modular forms and classical half-integral weight holomorphic modular forms. However, the approach used by Bringmann, Guerzhoy, and Kane only works for half-integral weight forms of weight $k + \frac{1}{2}$ where $k \geq 2$. This raises the natural question of whether a similar result holds for forms lower weights, in particular, the original forms developed by Zagier.

Yet, Zagier's pair of sets of forms are not alone in their interconnectedness. In [10], Folsom and Ono construct a startlingly similar grid: a set of weakly holomorphic modular forms of weight 3/2 which, when lined up term-by-term, form a set of mock modular forms of weight 1/2 in [10], the first of which is essentially Ramanujan's third-order mock theta-function

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

Guerzhoy follows up in [11] by proving that all the coefficients of these forms are rational numbers with bounded denominator. Zwegers then strengthens this result in [15] to prove that these coefficients are all integers.

Our goal in this paper is to obtain similar p-adic statements to those in [6] for the parallel cases of Zagier's forms and Folsom-Ono's forms.

Theorem 1.1. *Let $v_p(\cdot)$ be the p-adic valuation, normalized such that $v_p(p) = 1$, and for g a Fourier series with principal part $\sum_\alpha c_\alpha q^\alpha$ for a finite set of α, define $w(g) := \max\{\lfloor \frac{v_p(\alpha)}{2} \rfloor\}$. Then, assuming the definitions in §2.1, 2.2, and 2.3, the following are true.*

(1) *Let p be an odd prime. Suppose g is a weight $\frac{3}{2}$ weakly holomorphic modular form in $M_{\frac{3}{2}}^{!,+}(4)$ with integer coefficients. Then for $n \geq w(g)$, we have*

$$g|T\left(p^{2n+4}\right) - g|T\left(p^{2n}\right) \equiv 0 \quad (\mathrm{mod}\ p^{n-w(g)}).$$

(2) *Let $p \geq 5$ be a prime. Let G be a weight $\frac{3}{2}$ weakly holomorphic modular form in $M_{\frac{3}{2}}^{!,*}(144, \chi_{12})$ with integer coefficients. Then for $n \geq w(G)$, we have*

$$G|T_{12}\left(p^{2n+4}\right) - G|T_{12}\left(p^{2n}\right) \equiv 0 \quad (\mathrm{mod}\ p^{n-w(G)}).$$

Remark.

(1) Theorem 1.1 part (1) was proven independently in Ahlgren's "Hecke relations for traces of singular moduli," by similar methods [[1], Theorem 2]. We were not aware of this result at the time we submitted our paper.

(2) In [2], Ahlgren and Kim find similar relations to those in our Theorem 1.1 for other grids, namely, grids that involve the function $\mathrm{spt}(n)$, which counts the number of smallest parts in all partitions of n and other smallest parts functions. They also find a Hecke relation for the mock theta function $f(q)$.

Example. Let $p = 3$ and take Zagier's form g_4 (see §2.2 for the construction) with principal part q^{-4}, $v_3(4) = 0$, and let $n = 1$.

We have $g_4|T_{p^2}$ and $g_4|T_{p^6}$ (computed modulo 3^9, for convenience) as follows:

$$g_4|T_{p^2} \equiv 3q^{-36} + q^{-4} + 19675 + 19193q^3 + 6555q^4 + 13110q^7 + 9665q^8 + 4197q^{11}$$
$$+ 7517q^{12} + 8724q^{15} + 19665q^{16} + 13110q^{19} + O(q^{20}) \quad (\mathrm{mod}\ 3^9)$$

$$g_4|T_{p^6} \equiv 27q^{-2916} + 9q^{-324} + 3q^{-36} + q^{-4} + 19603 + 19679q^3 + 19677q^4 + 19671q^7$$
$$+ 9665q^8 + 4197q^{11} + 19667q^{12} + 19659q^{15} + 19665q^{16} + 19671q^{19} + O(q^{20}) \quad (\mathrm{mod}\ 3^9)$$

The minimum 3-adic valuation of the coefficients of $g_4|T_{p^6} - g_4|T_{p^2}$ (computed up to $O(q^{274})$) is 2, so $g_4|T_{p^6} - g_4|T_{p^2} \equiv 0 \pmod{3^2}$.

In [6], Bringmann, Guerzhoy, and Kane explore the close p-adic relationship between the operators U and T, exploiting the properties of the T-operator to yield a conclusion about the U-operator. To this end, we derive the following relation from the action of Hecke operators on our parallel sets of pairs of functions.

Theorem 1.2. *Assuming the same notation as in Theorem 1.1, for g_D (resp. G_D) the weight $\frac{3}{2}$ functions in Zagier's (resp. Folsom-Ono's) grid, the following are true. Let $j \in \mathbb{Z}_{\geq 0}$ with $p^2 \nmid j$. Then for $v, s \in \mathbb{Z}_{\geq 0}$.*

(1) *Denote $g_D = \sum_d b(D, d)q^d$, then for $i \in \mathbb{Z}$ such that $\left(\frac{-i}{p}\right) = \left(\frac{j}{p}\right)$, we have*

$$b\left(p^{2v}j, p^{2v+2s}i\right) \equiv 0 \pmod{p^s}.$$

(2) *Denote $G_D = \sum_d B(D, d)q^d$, then for $i \in \mathbb{Z}$ such that $\left(\frac{-i}{p}\right) = \left(\frac{j}{p}\right)$, we have*

$$B\left(p^{2v}j, p^{2v+2s}i\right) \equiv 0 \pmod{p^s}.$$

This paper is organized as follows: in §2.1 we define notation and recall preliminaries. Then, in §2.2 and §2.3, we describe Zagier's and Folsom-Ono's half-integral weight weakly holomorphic modular forms in terms of their expansions, their actions under Hecke operators, and their duality properties. Finally, in §3, we prove Theorems 1.1 and 1.2. Throughout this paper, p is taken to be a prime unless otherwise stated.

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2. NUTS AND BOLTS

In this section, we define Zagier's weight 1/2 and 3/2 forms, describe the way the Hecke operators act on them, and state their duality properties. We also give the analogous description for those forms that make up the Folsom-Ono grid. First, we define some notation concerning Hecke operators and the spaces of these forms.

2.1. Notation and Preliminaries.

The congruence subgroup $\Gamma_0(N)$ is defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

We call f a *weakly holomorphic modular form* of weight $k + \frac{1}{2}$ and level $4N$ and Nebentypus χ if it is a holomorphic function on the upper half-plane $\mathfrak{H}$ that satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)\left(\frac{c}{d}\right)^{2k+1} \epsilon_d^{-2k-1}(c\tau + d)^{k+\frac{1}{2}}f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, where, ϵ_d is either 1 or i depending on whether $d \equiv 1$ (mod 4) or $d \equiv 3$ (mod 4), and its poles, if any, are supported at the cusps (see [12]).

A *cusp form* is a modular form that vanishes at all cusps. Similarly, a *weakly holomorphic cusp form* is a weakly holomorphic modular form which has zero constant term at all cusps [12].

We denote the space of holomorphic (resp. weakly holomorphic) modular forms of weight $k + \frac{1}{2}$ and level N, with Nebentypus χ by $M_{k+\frac{1}{2}}(N, \chi)$ (resp. $M^!_{k+\frac{1}{2}}(N, \chi)$). We write $M^{!,+}_{k+\frac{1}{2}}(N, \chi)$ to emphasize when the forms in $M^!_{k+\frac{1}{2}}(N, \chi)$ are in Kohnen's plus space. Similarly, we write $S_{k+\frac{1}{2}}(N, \chi)$ (resp. $S^!_{k+\frac{1}{2}}(N, \chi)$) for the space of holomorphic (resp. weakly holomorphic) cusp forms of weight $k + \frac{1}{2}$ and level N with Nebentypus χ (and for weakly holomorphic cusp forms in the plus space, $S^{!,+}_{k+\frac{1}{2}}(N, \chi)$).

Let $M^{!,*}_{k+\frac{1}{2}}(N, \chi)$ signify the space spanned by Folsom-Ono's forms of weight $k + \frac{1}{2}$. That is, we denote by $M^{!,*}_k(N, \chi)$ the space of weakly holomorphic modular forms of weight k, level N, with Nebentypus χ whose coefficients are supported on exponents that are congruent to $(-1)^{k+1}$ (mod 24).

Furthermore, following the notation in [8], we define a *harmonic Maass form* on $\Gamma_0(4N)$ of weight $k + \frac{1}{2}$ as a smooth function $H : \mathfrak{H} \to \mathbb{C}$ such that

(1) For all matrices in $\Gamma_0(4N)$,

$$H\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right)^{2k+1} \epsilon_d^{-2k-1}(c\tau + d)^{k+\frac{1}{2}}H(\tau)$$

(2) $\Delta_{k+\frac{1}{2}}H = 0$ where Δ_k is the weight k hyperbolic Laplacian, defined as

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

for $\tau = x + iy \in \mathfrak{H}$ with $x, y \in \mathbb{R}$.

(3) There is a polynomial $P_H = \sum_{n \geq 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that

$$H(\tau) - P_H(\tau) = O(e^{-\epsilon y})$$

as $y \to +\infty$ for some $\epsilon > 0$, and analogous conditions are required at all cusps.

If H is a harmonic Maass form, there is a canonical splitting of H into

$$H = H^+ + H^-$$

where H^+ is the holomorphic part of H, called a *mock-modular form*, and H^- is the non-holomorphic part of H [13].

We also recall the classical cuspidal Poincaré series and the Maass-Poincaré series as in [13]: a general Poincaré series of weight k for $\Gamma_0(N)$ is given by

$$\mathbb{P}(m, k, N, \varphi_m; \tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_m^*|_k \gamma)(\tau),$$

where m is an integer, $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$ is the subgroup of translations in $\Gamma_0(N)$, and $\varphi_m^*(\tau) := \varphi_m(y)e^{2\pi i m x}$ for a function $\varphi_m : \mathbb{R}_{>0} \to \mathbb{C}$ which is $O(y^A)$ as $y \to 0$ for some $A \in \mathbb{R}$. We distinguish two special cases $(m > 0)$,

$$P(m, k, N; \tau) := \mathbb{P}(m, k, N, e^{-my}; \tau)$$
$$Q(-m, k, N; \tau) := \mathbb{P}(-m, 2-k, N, \mathcal{M}_{1-\frac{k}{2}}(-4\pi m y); \tau),$$

where $\mathcal{M}_s(y)$ is defined in terms of the M-Whittaker function. We often refer to $Q(-m, k, N; \tau)$ as a *Maass-Poincaré series*. It is well known that the Fourier expansions of the cuspidal Poincaré series are given by infinite sums of Kloosterman sums weighted by J-Bessel functions [13].

We also define the Petersson inner product for forms $f(\tau) \in M_k(N)$ and $g(\tau) \in M_k^!(N)$ (note that it also exists for $f(\tau) \in M_k(N)$ and $g(\tau)$ a harmonic Maass form), denoted $\langle f, g \rangle$, as the constant term in the expansion at $s = 0$ of the meromorphic continuation in s of the function

$$\frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \lim_{T \to \infty} \int_{\mathcal{F}_T(N)} f(\tau)\overline{g(\tau)}y^{k-s-2}dxdy$$

where

$$\mathcal{F}_T(N) := \bigcup_{\gamma \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} \gamma \mathcal{F}_T(SL_2(\mathbb{Z}))$$

and

$$\mathcal{F}_T(SL_2(\mathbb{Z})) = \{\tau \in \mathfrak{H} | \, |x| \leq \frac{1}{2}, |\tau| \geq 1, \text{and } y \leq T\}.$$

We define the differential operator by $D := \frac{1}{2\pi i}\frac{d}{dz}$ and let $\mathcal{H}_k(N)$ denote the space of harmonic Maass forms of weight k and level N.

Lemma 2.1. *(Theorem 7.8 in [13]) If $2 \leq k \in \mathbb{Z}$ the image of the map*

$$D^{k-1} : \mathcal{H}_{2-k}(N) \to M_k^!(N)$$

consists of forms in $M_k^!(N)$ which are orthogonal to cusp forms with respect to the regularized inner product, which also have constant term zero at all cusps of $\Gamma_0(N)$.

Let m be an integer and let $q := e^{2\pi i \tau}$. Following [6], we also recall, the standard $U(m)$ and $V(m)$ operators by their action on q-series $\sum a(n)q^n$ as follows,

$$\left(\sum a(n)q^n \right) |U(m) := \sum a(mn)q^n,$$
$$\left(\sum a(n)q^n \right) |V(m) := \sum a(n)q^{mn}.$$

Let $\chi(t,k)$, where t and k are integers, be the twisting operator, defined by

$$\left(\sum a(n)q^n \right) \otimes \chi(t,k) := \sum \left(\frac{(-1)^k \cdot n}{t} \right) a(n)q^n,$$

where $\left(\frac{\cdot}{\cdot} \right)$ is the standard Kronecker symbol. Note that all three of these operators preserve modularity, although they may change the level (see section 3.2 of [12]).

The weight $k + \frac{1}{2}$ Hecke operators are defined for f of level N and Nebentypus $\chi_s = \left(\frac{s}{\cdot} \right)$, where p is a prime and $p \nmid N$, by

$$(2.1) \qquad f|T_s\left(p^2\right) := f|U\left(p^2\right) + p^{k-1} \left(\frac{s}{p} \right) f \otimes \chi(p,k) + p^{2k-1} f|V\left(p^2\right),$$

We then define $T_s\left(p^{2m}\right)$ for $m \geq 1$ recursively by

$$(2.2) \qquad T_s\left(p^{2m}\right) := T_s\left(p^{2m-2}\right) T_s\left(p^2\right) - p^{2k-1} T_s\left(p^{2m-4}\right).$$

Furthermore, this can be extended to the formula

$$(2.3) \qquad T_s\left(p^{2m}\right) T_s\left(p^{2n}\right) = \sum_{t=0}^{m} p^t \cdot T_s\left(p^{2n+2m-4t}\right), \text{ for } m \leq n.$$

However, note that when we use the weight $\frac{1}{2}$ Hecke operator $T\left(p^2\right)$, we use the normalized version $p \cdot T\left(p^2\right)$ as in §6 of [14]. Also, in the case where the Nebentypus is the trivial character, as it is in Section 2.2, we suppress the s in the notation of the Hecke operator.

2.2. Zagier's Half-Integral Weight Modular Forms.

We define Zagier's forms f_d and g_D of weight $\frac{1}{2}$ and $\frac{3}{2}$, respectively. Following Zagier's notation, we label the forms f_d and g_D each according to the single term in their principal part, where the subscripts d and D are such that $f_d = q^{-d} + \sum_{D>0} a(D,d)q^D$ and $g_D = q^{-D} + \sum_{d\geq 0} b(D,d)q^d$. These forms are weakly holomorphic modular forms with integral coefficients on $\Gamma_0(4)$ in Kohnen's plus space which form a basis for $M_{\frac{1}{2}}^{!,+}(\Gamma_0(4))$ and $M_{\frac{3}{2}}^{!,+}(\Gamma_0(4))$, respectively (see [14]). First we define the following:

$$\theta(\tau) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \theta_1(\tau) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \quad g(\tau) := \theta_1(\tau)\frac{E_4(4\tau)}{\eta(4\tau)^6}.$$

The basis of $M_{\frac{1}{2}}^{!,+}(\Gamma_0(4))$, the f_d's, can be constructed as follows: $f_0 := \theta(\tau)$ and f_3 can be obtained from $[\theta(\tau), E_{10}(4\tau)]/\Delta(4\tau)$, which is a linear combination of f_0 and f_3. The rest of the f_d can be computed by multiplying f_{d-4} by $j(4\tau)$ and subtracting off multiples of the previously computed f_j ($0 \leq j < d$) so that f_d has the form $f_d = q^{-d} + O(q)$.

$$f_0 = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + O(q^{25})$$
$$f_3 = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 - 4096248q^9 + O(q^{12})$$
$$f_4 = q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + 51180012q^9 + O(q^{12})$$
$$\vdots$$

The construction for the basis of $M_{\frac{3}{2}}^{!,+}(\Gamma_0(4))$, the g_D's, is the same as that for the f_d's, except we start at $g_1 := g$ and construct g_4 as we constructed f_3 (with g in place of θ). In particular, the space of holomorphic modular forms of weight $3/2$ on $\Gamma_0(4)$ in the plus space is empty, and so there are no g_D's without a pole.

$$g_1 = q^{-1} - 2 + 248q^3 - 492q^4 + 4119q^7 - 7256q^8 + 33512q^{11} - 53008q^{12} + O(q^{15})$$
$$g_4 = q^{-4} - 2 - 26752q^3 - 143376q^4 - 8288256q^7 - 26124256q^8 + O(q^{11})$$
$$g_5 = q^{-5} + 0 + 85995q^3 - 565760q^4 + 52756480q^7 - 190356480q^8 + O(q^{11})$$
$$\vdots$$

A brief scan will show that the coefficients of g_D appear as the negatives of the D-th coefficients of f_d. In fact, this duality is true in general, as described in the following theorem from [14].

Theorem 2.2. *Let $f_d = q^{-d} + \sum_{D>0} a(D,d)q^D$ and $g_D = q^{-D} + \sum_{d\geq 0} b(D,d)q^d$, then*

(1) *For all $m \geq 0$, let $g_D|T(m^2) = \sum_d b_m(D,d)q^d$ and $f_d|T(m^2) = \sum_D a_m(D,d)q^D$. For all D and d such that $D \equiv 0,1 \pmod 4$ and $d \equiv 0,3 \pmod 4$,*

(2.4) $$a_m(D,d) = -b_m(D,d).$$

(2) *For any m, the following recursion holds*

$$a_m(1,d) = \sum_{n|m} n \cdot a\left(n^2, d\right).$$

The following proposition describes the action of the Hecke operators $T(p^{2n})$ on these g_D, extending the $T(p^2)$ case done in §6 of [14].

Proposition 2.3. *Let g_D be one of Zagier's weight 3/2 weakly holomorphic modular forms on $\Gamma_0(4)$. We write $D = p^{2v} \cdot j$, where $p^2 \nmid j$ and $v \geq 0$. Then, for all $0 \leq n < v$, we have*

$$g_D|T\left(p^{2n}\right) = \sum_{t=0}^{n} p^t \cdot g_{p^{2v-2n+4t} \cdot j},$$

and for all $n \geq v$, we have

(2.5) $$g_D|T\left(p^{2n}\right) = \sum_{t=0}^{n-v} \left(\frac{j}{p}\right)^{n-v-t} p^t \cdot g_{p^{2t}j} + \sum_{t=1}^{v} p^{n-v+t} \cdot g_{p^{2n-2v+4t} \cdot j}.$$

Proof. We first consider g_D where $D = j, p^2 \nmid j$. Recall that g_j looks like

$$g_j = q^{-j} + O(1).$$

Acting on g_j with the p^{2n}-th Hecke operator, we look just at the principal part. First, the base case of $T(p^2)$. From the definition in (2.1),

$$g_j|T\left(p^2\right) = g_j|U\left(p^2\right) + g_j \otimes \chi(p,1) + p \cdot g_j|V\left(p^2\right)$$

$$= p \cdot q^{-p^2 j} + \left(\frac{j}{p}\right)q^{-j} + O(1)$$

$$= pg_{p^2 j} + \left(\frac{j}{p}\right)g_j,$$

where the last line follows from the fact that Hecke operators do not change the level of the form, so $g_j|T(p^2)$ will remain in the space spanned by Zagier's forms. Since

these form a basis, any form in this space is determined entirely by its principal part. Proceeding inductively, assume that

$$(2.6) \qquad g_j | T\left(p^{2\ell}\right) = \sum_{t=0}^{\ell} \left(\frac{j}{p}\right)^{\ell-t} p^t \cdot g_{p^{2t}j}$$

holds for all $l \leq n$, for some $n \geq 0$. Then,

$$g_j | T\left(p^{2n+2}\right) = g_j | T\left(p^{2n}\right) T\left(p^2\right) - p \cdot g_j | T\left(p^{2n-2}\right)$$

$$(2.7) \qquad = \left(\sum_{t=0}^{n} \left(\frac{j}{p}\right)^{n-t} p^t \cdot g_{p^{2t}j}\right) | T\left(p^2\right) - \sum_{t=0}^{n-1} \left(\frac{j}{p}\right)^{n-t-1} p^{t+1} \cdot g_{p^{2t}j}$$

by applying (2.3), so we have that (2.7) condenses to

$$= \sum_{t=1}^{n} \left(\frac{j}{p}\right)^{n-t} p^t \cdot g_{p^{2t-2}\cdot j} + \left(\frac{j}{p}\right)^{n+1} g_j$$

$$= \sum_{t=0}^{n+1} \left(\frac{j}{p}\right)^{n-t+1} p^t \cdot g_{p^{2t}j}.$$

Note that this proves the proposition for $m = 0$ since then the second sum in the statement will be empty.

We now consider $D = p^{2v}j, v \geq 1$, where $p^2 \nmid j$. By (2.6), we can rewrite g_D as

$$g_D = \frac{g_j | T\left(p^{2v}\right) - \left(\frac{j}{p}\right) g_j | T\left(p^{2v-2}\right)}{p^v}.$$

Then, for $n \geq v$,

$$g_D | T\left(p^{2n}\right) = p^{-v}\left(g_j | T\left(p^{2v}\right) T\left(p^{2n}\right) - \left(\frac{j}{p}\right) g_j | T\left(p^{2v-2}\right) T\left(p^{2n}\right)\right)$$

$$= p^{-v} g_j | \left(\sum_{t=0}^{v} p^t \cdot T\left(p^{2n+2v-4t}\right) - \sum_{t=0}^{v-1} \left(\frac{j}{p}\right) p^t \cdot T\left(p^{2n+2v-4t-2}\right)\right)$$

again by applying (2.3), so we have

$$= g_j | T\left(p^{2n-2v}\right) + p^{-v} \sum_{t=1}^{v} p^{n+t} \cdot g_{p^{2n-2v+4t}\cdot j}$$

$$= \sum_{t=0}^{n-v} \left(\frac{j}{p}\right)^{n-v-t} p^t \cdot g_{p^{2t}j} + \sum_{t=1}^{v} p^{n-v+t} \cdot g_{p^{2n-2v+4t}\cdot j}.$$

The proof for $0 \leq n < v$ follows similarly, using the relation for $T\left(p^{2n}\right) T\left(p^{2v}\right)$ and summing over t from 0 to n, then reordering the sum. $\qquad\square$

Proposition 2.4. *Let f_d be one of Zagier's weight $1/2$ forms. We write $d = p^{2u} \cdot i$, where $p^2 \nmid i$ and $u \geq 0$. Then, for all $0 \leq n < u$, we have*

$$f_d | T\left(p^{2n}\right) = \sum_{t=0}^{n} p^{n-t} f_{p^{2u-2n+4t} \cdot i},$$

and for $n \geq u$, we have

$$(2.8) \qquad f_d | T\left(p^{2n}\right) = \sum_{t=0}^{n-u} \left(\frac{-i}{p}\right)^{n-u-t} p^u f_{p^{2t} i} + \sum_{t=1}^{u} p^{u-t} f_{p^{2n-2u+4t} \cdot i}.$$

Proof. The proof of this proposition follows in the same way as that of Proposition 2.3, with the exception that the weight $\frac{1}{2}$ Hecke operator is normalized as in §6 of [14], so we instead have

$$f_d | T\left(p^2\right) = p \cdot f_d | U\left(p^2\right) + f_d \otimes \chi(p, 1) + f_d | V\left(p^2\right).$$

$\qquad\square$

2.3. **Folsom-Ono Grid of Half-Integral Weight Forms.** In 3.5 of [10], Folsom and Ono introduce a grid of two sets of forms analogous to those of Zagier. Changing notation to avoid confusion with Zagier's f_d, g_D, let $\tilde{f}_m, \tilde{g}_m$ be the forms defined in Folsom and Ono's paper, using their notation with $m \in \mathbb{Z}$ as an index. We then define the sets F_d and G_D as follows, where d, D respectively each denote the integral power from the only term in the principal part:

$$F_{24m-23}(q) = q^{-1} \tilde{f}_m\left(q^{24}\right) \qquad\qquad G_{24m-1}(q) = -q \tilde{g}_m\left(q^{24}\right)$$
$$= -q^{-24m+23} + O\left(q^{23}\right), \qquad\qquad = q^{-24m+1} + O(q).$$

One set consists of weight $1/2$ mock modular forms F_d. Note that $F_1(q) = q^{-1} \tilde{f}_1(q^{24})$, where $\tilde{f}_1(q)$ is Ramanujan's third order mock-theta function

$$\tilde{f}_1(q) = f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

The F_d's have q-expansions as follows:

$$F_d(q) := -q^{-d} + \sum_{n=1}^{\infty} A(D, d) q^D.$$

Then, we have, in one direction,

$$\begin{aligned}
F_1(q) &= -q^{-1} + q^{23} - 2q^{47} + 3q^{71} - \cdots \\
F_{25}(q) &= -q^{-25} - 263q^{23} + 2781q^{47} - 17960q^{71} + \cdots \\
F_{49}(q) &= -q^{-49} + 3400q^{23} - 102060q^{47} + \cdots \\
F_{73}(q) &= -q^{-73} - 23374q^{23} + \cdots
\end{aligned}$$

Folsom and Ono prove that by grouping coefficients of the grid in columns, as in Zagier's case above, §3.3 of [10] formally defines a set of q-series G_D, which look like

$$G_D(q) := q^{-D} + \sum_{n=1}^{\infty} B(D, d)q^d.$$

For example, we have

$$\begin{aligned}
G_{23}(q) &= q^{-23} - q + 263q^{25} - 3400q^{49} + 23374q^{73} + \cdots \\
G_{47}(q) &= q^{-47} + 2q - 2781q^{25} + 102060q^{49} + \cdots \\
G_{71}(q) &= q^{-71} - 3q + 17960q^{25} + \cdots
\end{aligned}$$

These G_D are then proven to be weight $3/2$ weakly holomorphic modular forms; more precisely, they live in $M^!_{\frac{3}{2}}(\Gamma_0(144), \chi_{12})$. In analogy with Theorem 2.2, Folsom and Ono prove the following

Theorem 2.5. *For F_d and G_D as above, we have the relation*

(2.9) $$A(D, d) = -B(D, d).$$

As noted in the introduction, due to work of Guerzhoy in [11] and Zwegers in [15], these coefficients are integral.

Noting the similarity to the corresponding of the coefficients of the Zagier forms, we ask, as Zagier did, if the action of the Hecke operators preserves this duality. However, we first need to check that we can determine the action of the Hecke operators on G_D and F_d solely by looking at the principal part.

Lemma 2.6. *The space spanned by Folson-Ono's forms $G_D \in S^!_{\frac{3}{2}}(\Gamma_0(144), \chi_{12})$, supported on powers of q which are congruent to $1 \pmod{24}$, is invariant under the action of Hecke operators $T_{12}(p^{2n})$, where $p \nmid 144$, so the action of $T_{12}(p^{2n})$ on any form in that space is determined entirely by its principal part.*

Proof. First, let G be a form in the aforementioned space such that $G = G_{D_0}|T_{12}(p^{2n})$ for some D_0. Note that $D_0 = 24r+1$ for some $r \in \mathbb{Z}, r \geq 1$, and that G_{D_0} is supported only on coefficients equivalent to 1 (mod 24).

We first reexamine the Hecke operators under this Nebentypus to verify that G is supported on the correct arithmetic progression of exponents. We recall from (2.1) that,

$$T_{12}(p^2) = U(p^2) + \left(\frac{12}{p}\right) \chi(p, 1) + pV(p^2).$$

It is a well-known fact that for all $p \geq 5$, p prime, $p^2 \equiv 1$ (mod 24), so we know that the space of G_D where $D \equiv 1$ (mod 24) is Hecke-invariant, since any action of Hecke operators will change q-powers of n to a linear combination of q-powers $p^{2t}n \equiv n$ (mod 24). Thus, the resulting forms under Hecke transformations will also be supported on powers of q which are equivalent to 1 (mod 24).

Let us denote the principal part of G as $\displaystyle\sum_{\substack{n<0 \\ n\equiv 1 \ (\text{mod } 24)}} c_n q_n$. We can then construct a sum of G_D's to have the same principal part as G:

$$G' = \sum_{\substack{n<0 \\ n\equiv 1 \ (\text{mod } 24)}} c_n G_n.$$

Since the G_D are multiples of Poincaré series (see section 3.5 in [10] for the proof) constructed to have principal part zero at all other cusps but infinity, G' will have principal part zero at all other cusps as well. Furthermore, since the Hecke operator T_{12} cannot introduce a principal part, G will also have principal part zero at all other cusps. We also know, from [10], that the G_D have constant term zero at all cusps, so since T_{12} also cannot introduce a constant term, G also has constant term zero at all cusps. Therefore, we have that $G - G' \in S_{\frac{3}{2}}(\Gamma_0(144), \chi_{12})$, the space of holomorphic cusp forms with the same level and Nebentypus. If this space were empty, we would know that $G - G'$ would be zero, and as a result that G would be uniquely determined by its principal part. However, this space has dimension 2, and is generated by

$$B_1 = q - 5q^{25} + 7q^{49} - 11q^{121} + 13q^{169} + \cdots$$
$$B_2 = q^4 - 2q^{16} + 4q^{64} - 5q^{100} + \cdots$$

Thus, $G - G' = a_1 B_1$, since B_2 is supported on the wrong progression. However, we also claim that $a_1 = 0$, once again using the fact that the G_D are Poincaré series. Namely, we know by [3] that they are orthogonal to all cusp forms, i.e. $\langle G_D, g \rangle = 0$ for a cusp form g, where $\langle \cdot, \cdot \rangle$ denotes the regularized Petersson inner product (see

§2.1 and [5, 8, 13]). We know that for any $f \in M_k^!$ and $g \in S_k$,

$$\langle f | T_{12} \left(p^{2n} \right), g \rangle = \chi(n) \cdot \langle f, g | T_{12} \left(p^{2n} \right) \rangle.$$

Then, since $\langle G_D, g \rangle = 0$ for any cusp form g,

$$\langle G_D | T_{12} \left(p^{2n} \right), g \rangle = \chi(n) \langle G_D, g | T_{12} \left(p^{2n} \right) \rangle = 0,$$

for n coprime to the level, since the space of cusp forms is Hecke-invariant under $T_{12} \left(p^{2n} \right)$. From here, note that the Petersson inner product is linear in the first argument and some linear algebra will show that, in fact, $a_1 = 0$. We can extend this argument to $G = \left(\sum_D G_D \right) | T_{12} \left(p^{2n} \right)$ by noting that

$$G = \left(\sum_D G_D \right) | T_{12} \left(p^{2n} \right) = \sum_D \left(G_D | T_{12} \left(p^{2n} \right) \right),$$

so if we again form G' a sum of G_n's with the same principal part as G, then

$$\langle G - G', B_1 \rangle = \langle G, B_1 \rangle = \sum_D \langle G_D | T_{12} \left(p^{2n} \right), B_1 \rangle = 0.$$

Thus, the action of Hecke operators $T_{12} \left(p^{2n} \right)$ on the space spanned by Folsom-Ono's G_D's is entirely determined by the principal part. □

Lemma 2.7. *The space spanned by Folsom-Ono's forms $F_d \in S_{\frac{1}{2}}^! (\Gamma_0(144), \chi_{12})$, supported on powers of q which are congruent to $-1 \pmod{24}$, is invariant under the action of the normalized Hecke operators $T_{12} \left(p^{2n} \right)$, where $p \nmid 144$, so the action of $T_{12} \left(p^{2n} \right)$ on any form in that space is determined entirely by its principal part.*

Proof. As in the case of the weight $\frac{1}{2}$ Zagier forms, we use the normalized Hecke operators $T_{12} \left(p^2 \right) = p T_{\frac{1}{2}, 12} \left(p^2 \right)$. So by (2.1),

$$T_{12} \left(p^2 \right) = p U \left(p^2 \right) + \left(\frac{12}{p} \right) \chi(p, 0) + V \left(p^2 \right).$$

As in Lemma 2.6, for $p \geq 5$, p prime, we know that and $p^2 \equiv 1 \pmod{24}$, so the action of the Hecke operator will preserve the arithmetic progression of the exponents, which are all supported on q-powers equivalent to $-1 \pmod{24}$. Then, since Hecke operators are additively distributive, it suffices to consider F a form in the aforementioned space such that $F = F_{d_0} | T_{12} \left(p^{2n} \right)$. Using the normalization, F will have principal part with integral coefficients supported on q-powers $\equiv -1 \pmod{24}$, so we can construct a linear combination $F' = \sum_\alpha c_\alpha F_\alpha$ to have the same principal part as F. Now, we take the harmonic Maass form $H = H_{d_0} | T_{12} \left(p^{2n} \right)$ associated to F and take the harmonic Maass form H' associated to F'. We are interested in $F - F'$ so we first consider $H - H'$, a weight $\frac{1}{2}$ harmonic Maass form. In [11], Guerzhoy proves that such a harmonic Maass form will have a shadow which is a cusp form.

However, then, $H - H'$ has zero principal part at all cusps and thus must be 0, since every harmonic Maass form with nonzero shadow must have a pole at some cusp, which follows from Theorem 3.6 of [7]. Then, since Hecke operators respect the holomorphic and non-holomorphic parts of harmonic Maass forms, $F - F' = 0$, so the action of Hecke operators on individual F_d is entirely determined by principal part.

$\square$

Denote the coefficients under the action of the Hecke operators as $F_d|_{\frac{1}{2}} T\left(m^2\right) = \sum_D A_m(D, d)$ and $G_D|_{\frac{3}{2}} T\left(m^2\right) = \sum_d B_m(D, d)$.

Theorem 2.8. *For all $m, D, d \in \mathbb{Z}$,*

$$(2.10) \qquad A_m(D, d) = -B_m(D, d).$$

Proof. It suffices to prove the statement for $m = p$ prime. By the definition of the $3/2$ weight Hecke operator in (2.1),

$$B_p(D, d) = B(D, p^2 d) + \left(\frac{12}{p}\right)\left(\frac{-d}{p}\right) B(D, d) + pB(D, p^{-2}d).$$

If we consider $d < 0$, we have from the definition of the G_D that

$$B(D, d) = \delta_{-D, d},$$

where $\delta_{i,j}$ is the Kronecker δ−function. Thus,

$$B_p(D, d) = \delta_{-D, p^2 d} + \left(\frac{12}{p}\right)\left(\frac{-d}{p}\right) \delta_{-D, d} + p\delta_{-D, p^{-2}d}$$

$$= \delta_{-p^{-2}D, d} + \left(\frac{12}{p}\right)\left(\frac{D}{p}\right) \delta_{-D, d} + p\delta_{-p^2 D, d}.$$

Since the action of the Hecke operators on the G_D is determined entirely by their principal parts, by Lemma 2.6,

$$G_D|T_{12}\left(p^2\right) = G_{p^{-2}D} + \left(\frac{12}{p}\right)\left(\frac{D}{p}\right) G_D + pG_{p^2 D}.$$

Therefore, for all $D, d \in \mathbb{Z}$,

$$B_p(D, d) = B(p^{-2}D, d) + \left(\frac{12}{p}\right)\left(\frac{D}{p}\right) B(D, d) + pB(p^2 D, d)$$

$$= -A(p^{-2}D, d) - \left(\frac{12}{p}\right)\left(\frac{D}{p}\right) A(D, d) - pA(p^2 D, d)$$

by (2.9), so we have that

$$B_p(D, d) = -A_p(D, d)$$

by the definition of the Hecke operators for weight $1/2$ in (2.1). $\square$

Moreover, since these results appear so similar to those in the previous section, it is natural to ask the analogous questions answered in Propositions 2.3 and 2.4.

Proposition 2.9. *Let G_D be one of Folsom-Ono's weight 3/2 weakly holomorphic cusp forms on $\Gamma_0(144)$ with Nebentypus χ_{12}. We write $D = p^{2v} \cdot j, v \geq 0$, where $p^2 \nmid j$. Then, for all $0 \leq n < v$, we have*

$$G_D | T_{12}\left(p^{2n}\right) = \sum_{t=0}^{n} p^t \cdot G_{p^{2v-2n+4t} \cdot j},$$

and for all $n \geq v$, we have

$$(2.11) \qquad G_D | T_{12}\left(p^{2n}\right) = \sum_{t=0}^{n-v} \left(\frac{12j}{p}\right)^{n-v-t} p^t \cdot G_{p^{2t}j} + \sum_{t=1}^{v} p^{n-v+t} \cdot G_{p^{2n-2v+4t} \cdot j}.$$

Proposition 2.10. *Let F_d be one of Folsom-Ono's weight 1/2 mock modular forms on $\Gamma_0(144)$ with Nebentypus χ_{12}. We write $d = p^{2u} \cdot i$, where $p^2 \nmid i$ and $u \geq 0$. Then, for all $0 \leq n < u$,*

$$F_d | T_{12}\left(p^{2n}\right) = \sum_{t=0}^{n} p^{n-t} F_{p^{2n-2u+4t} \cdot i},$$

and for $n \geq u$,

$$(2.12) \qquad F_d | T_{12}\left(p^{2n}\right) = \sum_{t=0}^{n-u} \left(\frac{-12i}{p}\right)^{n-u-t} p^u f_{p^{2t}i} + \sum_{t=1}^{u} p^{u-t} F_{p^{2n-2u+4t} \cdot i}.$$

Proof of Proposition 2.9. Let G_D be one of Folsom-Ono's forms in [10]. Given Lemma 2.6, the proof of Proposition 2.9 follows similarly to that of Proposition 2.3, with the exception that a factor of $\left(\frac{12}{p}\right)$ appears along with each factor of $\left(\frac{j}{p}\right)$. $\square$

The proof of Proposition 2.10 is similarly analogous to that of Proposition 2.4.

3. Proofs of the Main Results

3.1. **Proof of Theorem 1.1.** We first prove two Propositions needed to prove Theorem 1.1.

Proposition 3.1. *Let g_D be one of Zagier's forms. We write $D = p^{2v} \cdot j$, where $v \geq 0$ and $p^2 \nmid j$. For $n \geq v$, we have*

$$(3.1) \qquad g_D | T \left(p^{2n+2}\right) - \left(\frac{j}{p}\right) \cdot g_D | T \left(p^{2n}\right) \equiv 0 \quad (\text{mod } p^{n-v+1}).$$

Proof. Suppose that $D = j, p^2 \nmid j$. Then, if we consider the difference of successive Hecke operators, by (2.5) of Proposition 2.3:

$$g_j | T \left(p^{2n+2}\right) - \left(\frac{j}{p}\right) g_j | T \left(p^{2n}\right) = \sum_{t=0}^{n+1} \left(\frac{j}{p}\right)^{n-t+1} p^t \cdot g_{p^{2t} \cdot j}$$

$$- \left(\frac{j}{p}\right) \sum_{t=0}^{n} \left(\frac{j}{p}\right)^{n-t} p^t \cdot g_{p^{2t} \cdot j}$$

$$= p^{n+1} g_{p^{2n+2} \cdot j}$$

$$\equiv 0 \quad (\text{mod } p^{n+1}).$$

We then treat the cases where $D = p^{2v} j, p^2 \nmid j$. Then, by Proposition 2.3, considering a difference of Hecke operators, we have, again by (2.5),

$$g_D | T \left(p^{2n+2}\right) - \left(\frac{j}{p}\right) g_D | T \left(p^{2n}\right) = g_j | T \left(p^{2n-2v+2}\right) + \sum_{t=1}^{v} p^{n-v+t+1} \cdot g_{p^{2n-2v+4t+2} \cdot j}$$

$$- \left(\frac{j}{p}\right) g_j | T \left(p^{2n-2v}\right) - \left(\frac{j}{p}\right) \sum_{t=1}^{v} p^{n-v+t} \cdot g_{p^{2n-2v+4t} \cdot j}$$

$$= p^{n-v+1} g_{p^{2n-2v+2} \cdot j}$$

$$+ p^{n-v+1} \sum_{t=0}^{v-1} \left(p^{t+1} g_{p^{2n-2v+4t+4} \cdot j} - \left(\frac{j}{p}\right) p^t g_{p^{2n-2v+4t+2} \cdot j}\right)$$

$$\equiv 0 \quad (\text{mod } p^{n-v+1}).$$

$\square$

Proposition 3.2. *Let G_D be one of Folsom-Ono's weakly holomorphic modular forms of weight $3/2$ on $\Gamma_0(144)$. We write $D = p^{2m} \cdot j$, where $m \geq 0$ and $p^2 \nmid j$. For n sufficiently large:*

$$(3.2) \qquad G_D | T_{12} \left(p^{2n+2}\right) - \left(\frac{12}{p}\right) \left(\frac{j}{p}\right) \cdot G_D | T_{12} \left(p^{2n}\right) \equiv 0 \quad (\text{mod } p^{n-m+1}).$$

Proof. The proof of Proposition 3.2 follows exactly as that of Proposition 3.1, with Proposition 2.9 taking the place of Proposition 2.3 and a factor of $\left(\frac{12}{p}\right)$ along with each factor of $\left(\frac{i}{p}\right)$. □

Proof of Theorem 1.1. By Zagier in [14], the space of weakly holomorphic modular forms in plus space has an integral basis of the g_D's considered in Proposition 3.1. Thus, we can express g as a sum of such g_D. Let

$$g = \sum_D c_D \cdot g_D.$$

If we express $D = p^{2v_D} \cdot j_D$ where $p^2 \nmid j_D$, let $w = \max\{v_D\}$ Then, for $n \geq w$,

$$g|T\left(p^{2n+4}\right) - g|T\left(p^{2n}\right) = \sum_D c_D \cdot \left(g_D|T\left(p^{2n+4}\right) - g_D|T\left(p^{2n}\right)\right)$$

$$= \sum_D c_D \cdot \left(g_D|T\left(p^{2n+4}\right) - \left(\frac{j_D}{p}\right) g_D|T\left(p^{2n+2}\right)\right.$$

$$\left. + \left(\frac{j_D}{p}\right) g_D|T\left(p^{2n+2}\right) - g_D|T\left(p^{2n}\right)\right)$$

$$= \sum_D c_D \cdot \left(g_D|T\left(p^{2n+4}\right) - \left(\frac{j_D}{p}\right) g_D|T\left(p^{2n+2}\right)\right)$$

$$+ \sum_{p\nmid j_D} c_D \left(\frac{j_D}{p}\right) \left(g_D|T\left(p^{2n+2}\right) - \left(\frac{j_D}{p}\right) g_D|T\left(p^{2n}\right)\right)$$

$$- \sum_{p\mid j_D} c_D \cdot g_D|T\left(p^{2n}\right).$$

Here, using Proposition 3.1, the first sum is equivalent to 0 (mod p^{n-w+2}), the second sum is equivalent to 0 (mod p^{n-w+1}), and the third sum is equivalent to 0 (mod p^{n-w}). Thus,

$$g|T\left(p^{2n+4}\right) - g|T\left(p^{2n}\right) \equiv 0 \pmod{p^{n-w}}$$

Given Proposition 3.2, the proof of Theorem 1.1 (2) follows exactly as does that of Theorem 1.1 (1) after Proposition 3.1, with the exception that a factor of $\left(\frac{12}{p}\right)$ appears along with each factor of $\left(\frac{i}{p}\right)$.

3.2. **Proof of Theorem 1.2.** We first prove two Lemmas needed to prove Theorem 1.2.

Lemma 3.3. *For $n \geq v$, we have that*

$$(3.3) \qquad b_{p^n}\left(p^{2v}j,d\right) - \left(\frac{j}{p}\right) b_{p^{n-1}}\left(p^{2v}j,d\right) \equiv 0 \pmod{p^{n-v}}$$

and

$$(3.4) \qquad B_{p^n}\left(p^{2v}j,d\right) - \left(\frac{12}{p}\right)\left(\frac{j}{p}\right) B_{p^{n-1}}\left(p^{2v}j,d\right) \equiv 0 \pmod{p^{n-v}}$$

Proof. Recalling that we let the coefficients of g_D, G_D after the action of the Hecke operator be denoted as

$$g_D|T\left(p^{2n}\right) = q^{-D} + \sum_d b_{p^n}(D,d)q^d$$

$$G_D|T_{12}\left(p^{2n}\right) = q^{-D} + \sum_d B_{p^n}(D,d)q^d,$$

we equate the coefficients of the q^d-th powers on either side of the equations (3.1) and (3.2) in Propositions 3.1 and 3.2, respectively.   □

Lemma 3.4. *For $n \geq u$,*

$$(3.5)$$

$$a_{p^n}\left(D,p^{2u}i\right) = \sum_{t=0}^{n-u}\left(\frac{-i}{p}\right)^{n-u-t} p^u a\left(D,p^{2t}i\right) + \sum_{t=1}^{u} p^{u-t} a\left(D,p^{2n-2u+4t}\cdot i\right),$$

and

$$(3.6)$$

$$A_{p^n}\left(D,p^{2u}i\right) = \sum_{t=0}^{n-u}\left(\frac{12}{p}\right)^{n-u-t}\left(\frac{-i}{p}\right)^{n-u-t} p^u A\left(D,p^{2t}i\right) + \sum_{t=1}^{u} p^{u-t} A\left(D,p^{2n-2u+4t}\cdot i\right).$$

Proof. Recalling that we let the coefficients of f_d after the action of the Hecke operator be denoted as

$$f_d|T\left(p^{2n}\right) = \sum_D a_{p^n}(D,d)q^D$$

$$F_d|T_{12}\left(p^{2n}\right) = \sum_D A_{p^n}(D,d)q^D,$$

we equate the coefficients of the q^D-th powers on either side of (2.8) and (2.12) in Propositions 2.4 and 2.10, respectively.   □

Proof of Theorem 1.2. If we let $u = 0$ in (3.5), we obtain

$$a_{p^n}(D, i) = \sum_{t=0}^{n} \left(\frac{-i}{p} \right)^{n-t} b\left(D, p^{2t} i \right).$$

However, we also know, by (2.4) of Theorem 2.2 that

$$a_m(D, d) = -b_m(D, d)$$

for all m, D, d. Then, substituting into (3.3) of Lemma 3.3,

$$b_{p^n}\left(p^{2v} j, i \right) - \left(\frac{j}{p} \right) b_{p^{n-1}}\left(p^{2v} j, i \right) \equiv 0 \quad (\text{mod } p^{n-v})$$

$$-a_{p^n}\left(p^{2v} j, i \right) + \left(\frac{j}{p} \right) a_{p^{n-1}}\left(p^{2v} j, i \right) \equiv 0 \quad (\text{mod } p^{n-v})$$

$$-\sum_{t=0}^{n} \left(\frac{-i}{p} \right)^{n-t} a\left(p^{2v} j, p^{2t} i \right) + \left(\frac{j}{p} \right) \sum_{t=0}^{n-1} \left(\frac{-i}{p} \right)^{n-t-1} a\left(p^{2v} j, p^{2t} i \right) \equiv 0 \quad (\text{mod } p^{n-v})$$

If $\left(\frac{j}{p} \right) = \left(\frac{-i}{p} \right)$, then the terms for $0 \leq t \leq n-1$ cancel between sums and

$$-a\left(p^{2v} j, p^{2n} i \right) \equiv 0 \quad (\text{mod } p^{n-v}).$$

So, applying Theorem 2.2 again,

$$b\left(p^{2v} j, p^{2n} i \right) \equiv 0 \quad (\text{mod } p^{n-v}).$$

In particular, when we then substitute $n = v + s$,

$$b\left(p^{2v} j, p^{2v+2s} i \right) \equiv 0 \quad (\text{mod } p^s).$$

$\square$

As in previous sections, this proof follows exactly as that of Theorem 1.2 (1), with the exception that a factor of $\left(\frac{12}{p} \right)$ appears with every factor of $\left(\frac{-i}{p} \right)$.

REFERENCES

[1] S. Ahlgren. Hecke relations for traces of singular moduli. *Bull. Lond. Math. Soc.*, 44(1):99–105, 2012.
[2] S. Ahlgren and B. Kim. Mock modular grids and Hecke relations for mock modular forms. *Forum Math.*, 26(4):1261–1287, 2014.
[3] C. Alfes. *CM values and Fourier coefficients of harmonic Maass forms.* PhD thesis, Technische Universität Darmstadt, 2015.
[4] R. Borcherds. Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products. *Invent. Math.*, 120:161–213, 1995.

[5] R. Borcherds. Automorphic forms with singularities on grassmannians. *Invent. Math.*, 132:491–562, 1998.

[6] K. Bringmann, P. Guerzhoy, and B. Kane. Half-integral weightf p-adic coupling in weakly holomorphic and holomorphic modular forms. preprint, 2015.

[7] J. H. Bruinier and J. Funke. On two geometric theta lifts. *Duke Math J.*, 125:45–90, 2004.

[8] J. H. Bruinier, K. Ono, and R. Rhoades. Differential operators for harmonic weak maass forms and the vanishing of hecke eigenalues. *Math. Ann.*, 342:673–693, 2008.

[9] W. Duke and P. Jenkins. Integral traces of singular values of weak maass forms. *Algebra and Number Theory*, 2:573–593, 2008.

[10] A. Folsom and K. Ono. Duality involving the mock theta function f(q). *J. Lond. Math. Soc. (2)*, 77:320–334, 2008.

[11] P. Guerzhoy. Rationality of the Folsom-Ono grid. *Proc. Amer. Math. Soc.*, 137(no. 5):1569–1577, 2009.

[12] K. Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.

[13] K. Ono. Unearthing the visions of a master: harmonic Maass forms and number theory. In *Current developments in mathematics, 2008*, pages 347–454. Int. Press, Somerville, MA, 2009.

[14] D. Zagier. *Motives, polylogarithms, and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, chapter Traces of Singular Moduli, pages 211–244. Int. Press, Somerville, MA, 2002.

[15] S. Zwegers. The Folsom-Ono grid contains only integers. *Proc. Amer. Math. Soc.*, 137(no. 5):1579–1584, 2009.

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